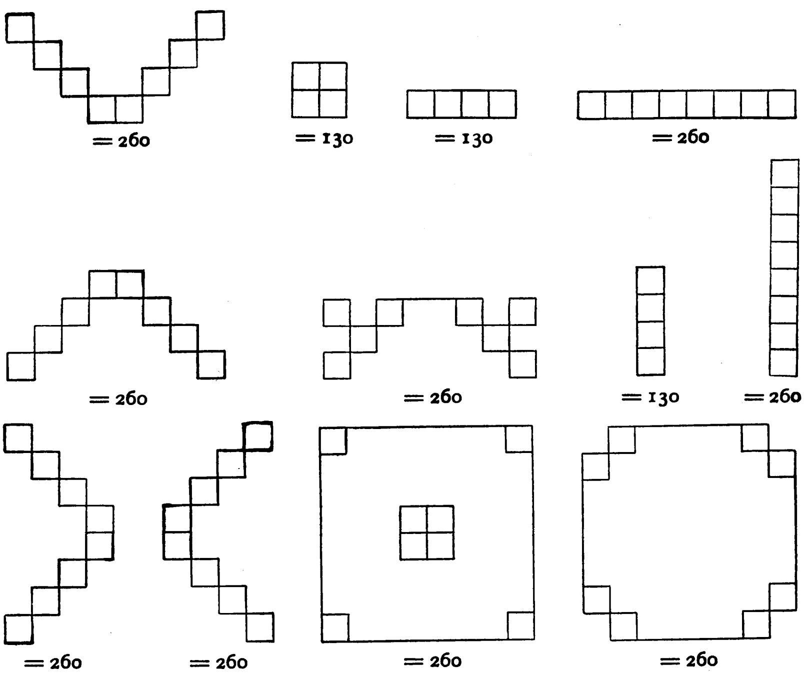
CRITICISMS AND DISCUSSIONS.

THE FRANKLIN SQUARES AND OTHER MATHEMATICAL DIVERSIONS.

The July number of *The Monist* contained an article by C. A. Browne, of New Orleans, on "Magic Squares and Pythagorean



PROPERTIES OF FRANKLIN'S 8×8 SQUARE.

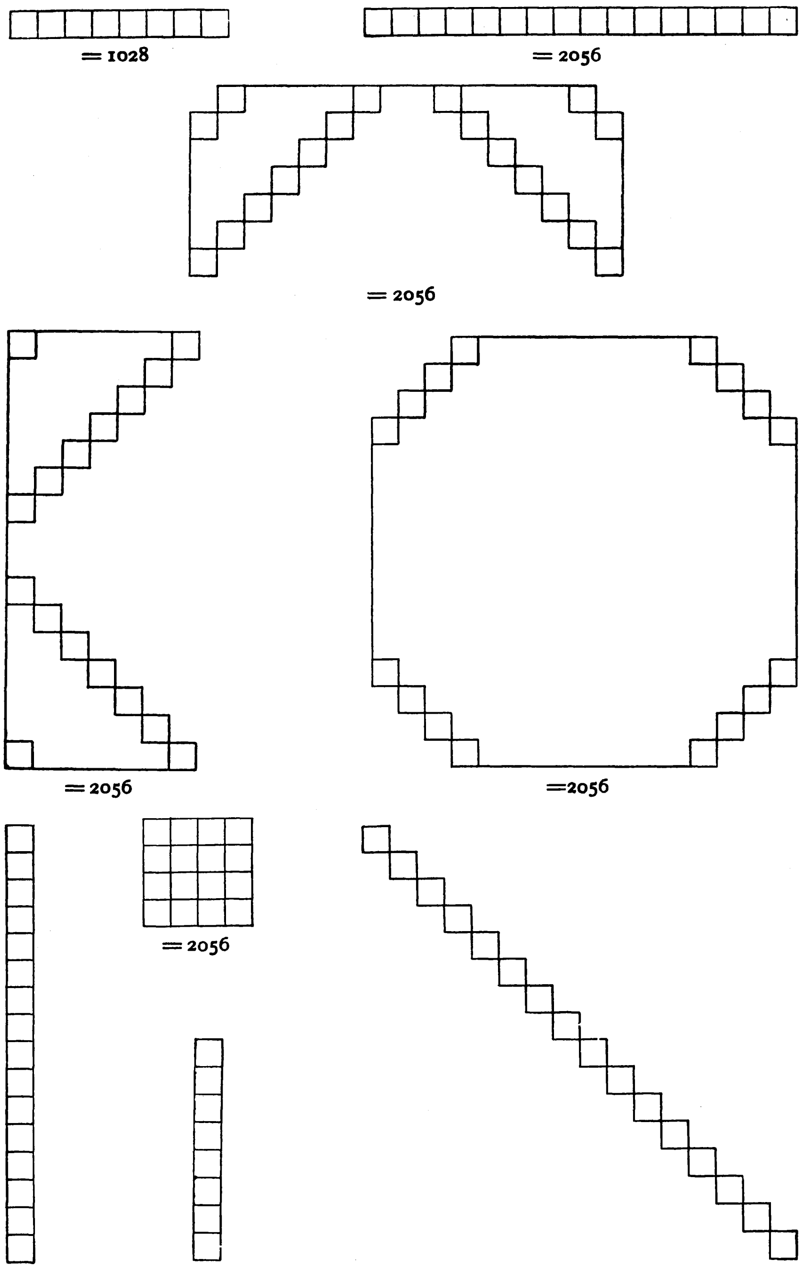
Numbers," in which he referred to Benjamin Franklin's interest in the subject. Upon inquiry I found in *The Life and Times of Benjamin Franklin*, by James Parton, (Vol. I, pp. 255-257), a brief

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

FRANKLIN 8×8 SQUARE.

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

FRANKLIN 16×16 SQUARE.

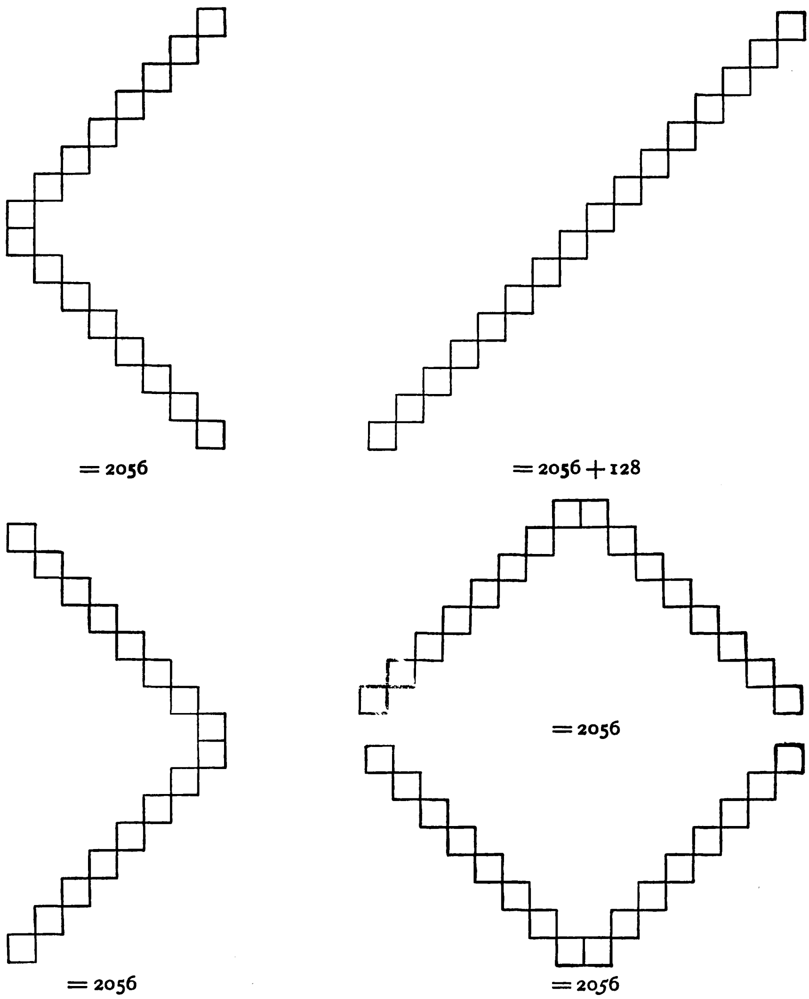


PROPERTIES OF FRANKLIN'S 16×16 SQUARE.

account of two magic squares, one 8×8, the other 16×16, both of which are here reproduced.

Mr. Parton explains the 8×8 square as follows:

"This square, as explained by its contriver, contains astonishing



PROPERTIES OF FRANKLIN'S 16×16 SQUARE.
(Continued from preceding page.)

"properties: every straight row (horizontal or vertical) added together makes 260, and each half row half 260. The bent row of "eight numbers ascending and descending diagonally, viz., from 16

"ascending to 10, and from 23 descending to 17, and every one of "its parallel bent rows of eight numbers, makes 260. Also, the bent "row from 52 descending to 54, and from 43 ascending to 45, and "every one of its parallel bent rows of eight numbers, makes 260. "Also, the bent row from 45 to 43, descending to the left, and from "23 to 17, descending to the right, and every one of its parallel bent "rows of eight numbers, makes 260. Also, the bent row from 52 "to 54, descending to the right, and from 10 to 16, descending to "the left, and every one of its parallel bent rows of eight numbers, "makes 260. Also, the parallel bent rows next to the above-men-"tioned, which are shortened to three numbers ascending and three "descending, etc., as from 53 to 4 ascending and from 29 to 44 "descending, make, with the two corner numbers, 260. Also, the two "numbers, 14, 61, ascending, and 36, 19, descending, with the lower "four numbers situated like them, viz., 50, 1, descending, and 32, 47, "ascending, makes 260. And, lastly, the four corner numbers, with "the four middle numbers, make 260.

"But even these are not all the properties of this marvelous "square. Its contriver declared that it has 'five other curious ones,' "which he does not explain; but which the ingenious reader may "discover if he can."

These remarkable characteristics which Mr. Parton enumerates are illustrated graphically in the accompanying diagrams in which the relative position of the cells containing the numbers which make up the number 260, is indicated by the relation of the small hollow squares.

Franklin's 16×16 square is constructed upon the same principle as the smaller, and Mr. Parton continues:

"Nor was this the most wonderful of Franklin's magical "squares. He made one of sixteen cells in each row, which besides "possessing the properties of the squares given above (the amount, "however added, being always 2056), had also this most remark-"able peculiarity: a square hole being cut in a piece of paper of such "a size as to take in and show through it just sixteen of the little "squares, when laid on the greater square, the sum of sixteen num-"bers, so appearing through the hole, wherever it was placed on the "greater square, should likewise make 2056."

The additional peculiarity which Mr. Parton notes of the 16× 16 square is no more remarkable than the corresponding fact which is true of the smaller square, that the sum of the numbers in any

	1	2	3	4	5	6	7	8	
A	1	2	3	4	5	6	7	8	A
B	9	10	11	12	13	14	15	16	B
C	17	18	19	20	21	22	23	24	C
D	25	26	27	28	29	30	31	32	D
E	33	34	35	36	37	38	39	40	E
F	41	42	43	44	45	46	47	48	F
G	49	50	51	52	53	54	55	56	G
H	57	58	59	60	61	62	63	64	H
	1	2	3	4	5	6	7	8	

THE PLAN OF CONSTRUCTION.

A_1	B_8	C	D	E	F	G	H
H_7	G_2	F	E	D	C	B	A
3	6						
5	4						
5	4						
3	6						
7	2						
1	8						

First Step.
KEY TO THE SCHEME OF SIMPLE ALTERNATION-

A_1	B_8	C_1	D_8	E_1	F_8	G_1	H_8
H_7	G_2	F_7	E_2	D_7	C_2	B_7	A_2
A_3	B_6	C_3	D_6	E_3	F_6	G_3	H_6
H_5	G_4	F_5	E_4	D_5	C_4	B_5	A_4
A_5	B_4	C_5	D_4	E_5	F_4	G_5	H_4
H_3	G_6	F_3	E_6	D_3	C_6	B_3	A_6
A_7	B_2	C_7	D_2	E_7	F_2	G_7	H_2
H_1	G_8	F_1	E_8	D_1	C_8	B_1	A_8

Second Step.
COMPLETED SCHEME OF SIMPLE ALTERNATION.

1	16	17	32	33	48	49	64
63	50	47	34	31	18	15	2
3	14	19	30	35	46	51	62
61	52	45	36	29	20	13	4
5	12	21	28	37	44	53	60
59	54	43	38	27	22	11	6
7	10	23	26	39	42	55	58
57	56	41	40	25	24	9	8

Third step.

8×8 MAGIC SQUARE CONSTRUCTED BY SIMPLE ALTERNATION.

2×2 combination of its cells yields 130. The properties of the larger square are also graphically represented here.

Franklin's squares were not known to me when I wrote my comments in the January number of *The Monist;* but at the first glance it became obvious that they belong to the same class as the Jaina square quoted by Professor David Eugene Smith (see *Monist* for January, 1906, p. 134), and that the clue to their construction must be sought in the same way.

We write down the numbers in numerical order and call the cells after the precedent of the chess-board, with two sets of symbols, letters and numbers. We call this "the plan of construction."

Before we construct the general scheme of Franklin's square we will build up another magic square, a little less complex in principle, which will be preparatory work for more complicated squares. We will simply intermix the ordinary series of numbers according to a definite rule alternately reversing the letters so that the odd rows are in alphabetical order and the even ones reversed. In order to distribute the numbers in a regular fashion so that no combination of letter and number would occur twice, we start with 1 in the upper left-hand corner and pass consecutively downwards, alternating between the first and second cells in the successive rows, thence ascending by the same method of simple alternation from 1 in the lower left-hand corner. We have now the key to a scheme for the distribution of numbers in an 8×8 magic square. It is the first step in the construction of the Franklin 8×8 magic square, and we call it "the key to the scheme of simple alternation."

It goes without saying that the effect would be the same if we begin in the same way in the right-hand corners,—only we must beware of a distribution that would occasion repetitions.

To complete the scheme we have to repeat the letters, alternately inverting their order row after row, and the first two given figures must be repeated throughout every row, as they are started. The top and bottom rows will read 1, 8; 1, 8; 1, 8; 1, 8. The second row from the top and also from the bottom will be 7, 2; 7, 2; 7, 2; 7, 2. The third row from the top and bottom will be 3, 6; 3, 6; 3, 6; 3, 6; and the two center rows 5, 4; 5, 4; 5. 4; 5, 4. In every line the sum of two consecutive figures yields 9. This is the second step, yielding the completed scheme of simple alternation.

The square is now produced by substituting for the letter and figure combinations, the corresponding figures according to the consecutive arrangement in the plan of construction.

Trying the results we find that all horizontal rows sum up to 260, while the vertical rows are alternately $260-4$, and $260+4$. The diagonal from the upper right to the lower left corner yields a sum of $260+32$, while the other diagonal from the left upper corner descending to the right lower corner makes $260-32$. The upper halves of the two diagonals yield 260, and also the sum of the lower halves, and the sum total of both diagonals is accordingly 520 or 2×260. The sum of the two left-hand half diagonals results in $260-16$, and the sum of the two half diagonals to the right-hand side makes $260+16$. The sum of the four central cells plus the four extreme corner cells yields also 260.

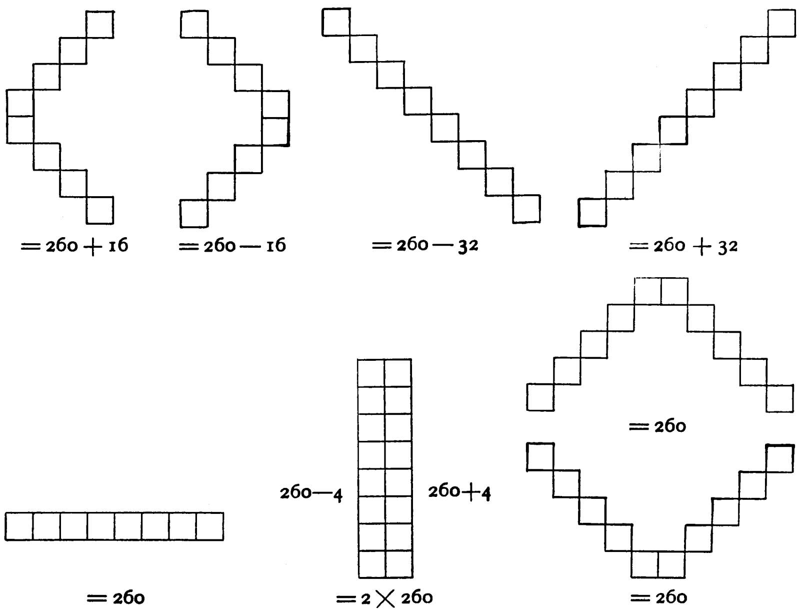
Considering the fact that the figures 1 to 8 of our scheme run up and down in alternate succession, we naturally have an arrangement of figures in which sets of two belong together. This binate peculiarity is evidenced in the result just stated, that the rows yield sums which are the same with an alternate addition and subtraction of an equal amount. So we have a symmetry which is astonishing and might be deemed magical, if it were not a matter of intrinsic necessity.

We represent these peculiarities in the adjoined diagrams which, however, by no means exhaust all the possibilities.

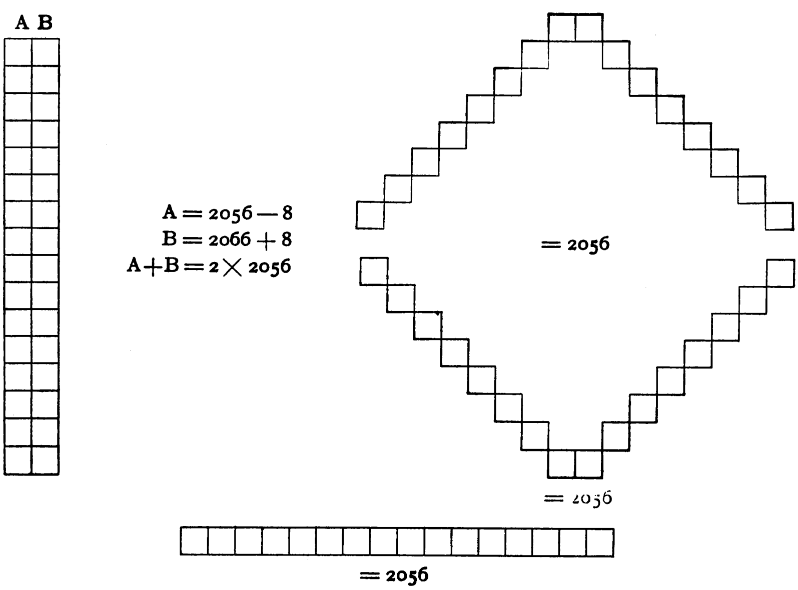
We must bear in mind that these magic squares are to be regarded as continuous; that is to say, they are as if their opposite sides in either direction passed over into one another as if they were joined both ways in the shape of a cylinder. In other words when we cross the boundary of the square on the right hand, the first row of cells outside to the right has to be regarded as identical with the first row of cells on the left; and in the same way the uppermost or first horizontal row of cells corresponds to the first row of cells below the bottom row. This remarkable property of the square will bring out some additional peculiarities which mathematicians may easily derive according to general principles; especially what was stated of the sum of the lower and upper half-diagonal of any bent series of cells running staircase fashion either upward or downward to the center, and hence proceeding in the opposite way to the other side.

The magic square constructed according to the method of simple alternation of figures is not, however, the square of Benjamin Franklin, but we can easily transform the former into the latter by slight modifications.

We notice that in certain features the sum total of the bent



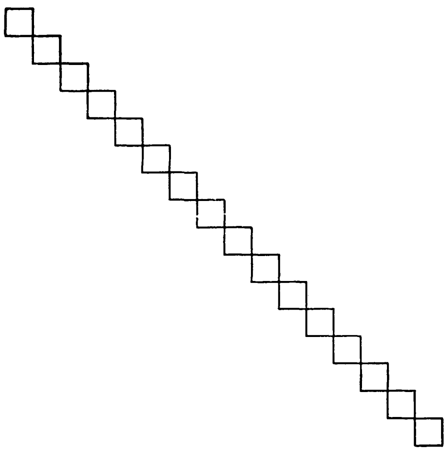
PROPERTIES OF 8×8 SQUARE BY SIMPLE ALTERNATION.



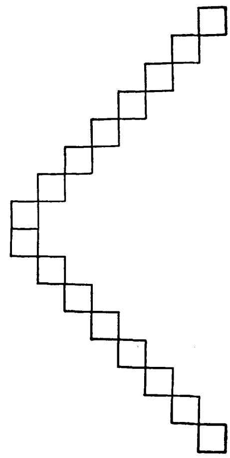
PROPERTIES OF 16×16 SQUARE BY SIMPLE ALTERNATION.

diagonals represents regularities which counterbalance one another on the right- and the left-hand side. In order to offset these results we have to shift the figures of our scheme.

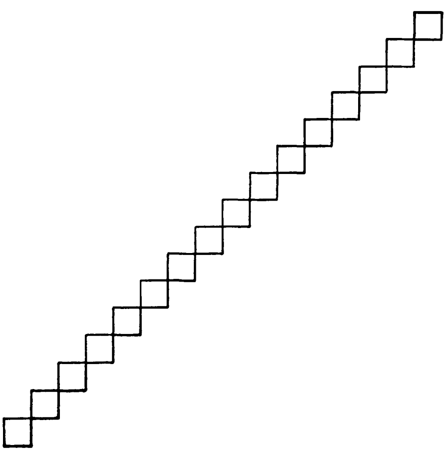
We take the diagram which forms the key to the scheme of our



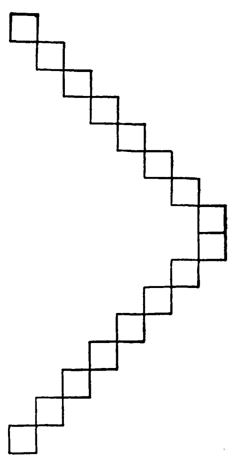
= 2056 — 128



= alternately 2056 — 64 and 2056 + 64



= 2056 + 128



= alternately 2056 — 64 and 2056 + 64

PROPERTIES OF 16×16 SQUARE BY SIMPLE ALTERNATION.
(Continued from preceding page.)

distribution by simple alternation, and cutting it in the middle, turn the lower half upside down, giving the first two rows as seen in the diagram in which the heavy lines indicate the cutting. Cutting then the upper half in two (i. e., in binate sections), and transposing

1	8
7	2
3	6
5	4
8	1
2	7
6	3
4	5

A_1	B_8	C	D	E	F	G	H
H_7	G_2	F	E	D	C	B	A
8	1						
2	7						
6	3						
4	5						
3	6						
5	4						

First Steps.

KEY TO THE SCHEME OF ALTERNATION WITH BINATE TRANSPOSITION.

A_1	B_8	C_1	D_8	E_1	F_8	G_1	H_8
H_7	G_2	F_7	E_2	D_7	C_2	B_7	A_2
A_8	B_1	C_8	D_1	E_8	F_1	G_8	H_1
H_2	G_7	F_2	E_7	D_2	C_7	B_2	A_7
A_6	B_3	C_6	D_3	E_6	F_3	G_6	H_3
H_4	G_5	F_4	E_5	D_4	C_5	B_4	A_5
A_3	B_6	C_3	D_6	E_3	F_6	G_3	H_6
H_5	G_4	F_5	E_4	D_5	C_4	B_5	A_4

Second Step.

SCHEME OF ALTERNATION WITH BINATE TRANSPOSITION.

1	16	17	32	33	48	49	64
63	50	47	34	31	18	15	2
8	9	24	25	40	41	56	57
58	55	42	39	26	23	10	7
6	11	22	27	38	43	54	59
60	53	44	37	28	21	12	5
3	14	19	30	35	46	51	62
61	52	45	36	29	20	13	4

Third Step.

SQUARE CONSTRUCTED BY ALTERNATION WITH BINATE TRANSPOSITION

G_4	H_5	A_4	B_5	C_4	D_5	E_4	F_5
B_6	A_3	H_6	G_3	F_6	E_3	D_6	C_3
G_5	H_4	A_5	B_4	C_5	D_4	E_5	F_4
B_3	A_6	H_3	G_6	F_3	E_6	D_3	C_6
G_7	H_2	A_7	B_2	C_7	D_2	E_7	F_2
B_1	A_8	H_1	G_8	F_1	E_8	D_1	C_8
G_2	H_7	A_2	B_7	C_2	D_7	E_2	F_7
B_8	A_1	H_8	G_1	F_8	E_1	D_8	C_1

SCHEME OF FRANKLIN'S 8×8 SQUARE.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
A	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	A
B	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	B
C	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	C
D	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	D
E	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	E
F	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	F
G	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	G
H	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127	128	H
I	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	I
K	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	K
L	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	L
M	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191	192	M
N	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	N
O	209	210	211	212	213	214	215	216	217	218	219	220	221	222	223	224	O
P	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240	P
Q	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	Q
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	

CONSECUTIVE ARRANGEMENT OF NUMBERS IN A 16×16 SQUARE.

A_{1}	B_{16}	C	D	E	F	G	H	I	K	L	M	N	O	P	Q
Q_{15}	P_{2}	O	N	M	L	K	I	H	G	F	E	D	C	B	A
3	14														
13	4														
5	12														
11	6														
7	10														
9	8														
9	8														
7	10														
11	6														
5	12														
13	4														
3	14														
15	2														
1	16														

KEY TO THE SCHEME OF SIMPLE ALTERNATION.

A_1	B_{16}	C_1	D_{16}	E_1	F_{16}	G_1	H_{16}	I_1	K_{16}	L_1	M_{16}	N_1	O_{16}	P_1	Q_{16}
Q_{15}	P_2	O_{15}	N_2	M_{15}	L_2	K_{15}	I_2	H_{15}	G_2	F_{15}	E_2	D_{15}	C_2	B_{15}	A_2
A_3	B_{14}	C_3	D_{14}	E_3	F_{14}	G_3	H_{14}	I_3	K_{14}	L_3	M_{14}	N_3	O_{14}	P_3	Q_{14}
Q_{13}	P_4	O_{13}	N_4	M_{13}	L_4	K_{13}	I_4	H_{13}	G_4	F_{13}	E_4	D_{13}	C_4	B_{13}	A_4
A_5	B_{12}	C_5	D_{12}	E_5	F_{12}	G_5	H_{12}	I_5	K_{12}	L_5	M_{12}	N_5	O_{12}	P_5	Q_{12}
Q_{11}	P_6	O_{11}	N_6	M_{11}	L_6	K_{11}	I_6	H_{11}	G_6	F_{11}	E_6	D_{11}	C_6	B_{11}	A_6
A_7	B_{10}	C_7	D_{10}	E_7	F_{10}	G_7	H_{10}	I_7	K_{10}	L_7	M_{10}	N_7	O_{10}	P_7	Q_{10}
Q_9	P_8	O_9	N_8	M_9	L_8	K_9	I_8	H_9	G_8	F_9	E_8	D_9	C_8	B_9	A_8
A_9	B_8	C_9	D_8	E_9	F_8	G_9	H_8	I_9	K_8	L_9	M_8	N_9	O_8	P_9	Q_8
Q_7	P_{10}	O_7	N_{10}	M_7	L_{10}	K_7	I_{10}	H_7	G_{10}	F_7	E_{10}	D_7	C_{10}	B_7	A_{10}
A_{11}	B_6	C_{11}	D_6	E_{11}	F_6	G_{11}	H_6	I_{11}	K_6	L_{11}	M_6	N_{11}	O_6	P_{11}	Q_6
Q_5	P_{12}	O_5	N_{12}	M_5	L_{12}	K_5	I_{12}	H_5	G_{12}	F_5	E_{12}	D_5	C_{12}	B_5	A_{12}
A_{13}	B_4	C_{13}	D_4	E_{13}	F_4	G_{13}	H_4	I_{13}	K_4	L_{13}	M_4	N_{13}	O_4	P_{13}	Q_4
Q_3	P_{14}	O_3	N_{14}	M_3	L_{14}	K_3	I_{14}	H_3	G_{14}	F_3	E_{14}	D_3	C_{14}	B_3	A_{14}
A_{15}	B_2	C_{15}	D_2	E_{15}	F_2	G_{15}	H_2	I_{15}	K_2	L_{15}	M_2	N_{15}	O_2	P_{15}	Q_2
Q_1	P_{16}	O_1	N_{16}	M_1	L_{16}	K_1	I_{16}	H_1	G_{16}	F_1	E_{16}	D_1	C_{16}	B_1	A_{16}

SCHEME OF SIMPLE ALTERNATION.

1	32	33	64	65	96	97	128	129	160	161	192	193	224	225	256
255	226	223	194	191	162	159	130	127	98	95	66	63	34	31	2
3	30	35	62	67	94	99	126	131	158	163	190	195	222	227	254
253	228	221	196	189	164	157	132	125	100	93	68	61	36	29	4
5	28	37	60	69	92	101	124	133	156	165	188	197	220	229	252
251	230	219	198	187	166	155	134	123	102	91	70	59	38	27	6
7	26	39	58	71	90	103	122	135	154	167	186	199	218	231	250
249	232	217	200	185	168	153	136	121	104	89	72	57	40	25	8
9	24	41	56	73	88	105	120	137	152	169	184	201	216	233	248
247	234	215	202	183	170	151	138	119	106	87	74	55	42	23	10
11	22	43	54	75	86	107	118	139	150	171	182	203	214	235	246
245	236	213	204	181	172	149	140	117	108	85	76	53	44	21	12
13	20	45	52	77	84	109	116	141	148	173	180	205	212	237	244
243	238	211	206	179	174	147	142	115	110	83	78	51	46	19	14
15	18	47	50	79	82	111	114	143	146	175	178	207	210	239	242
241	240	209	208	177	176	145	144	113	112	81	80	49	48	17	16

16×16 MAGIC SQUARE CONSTRUCTED BY SIMPLE ALTERNATION.

A_1	B_{16}	C	D	E	F	G	H	I	K	L	M	N	O	P	Q
Q_{15}	P_2	O	N	M	L	K	I	H	G	F	E	D	C	B	A
3	14														
13	4														
16	1														
2	15														
14	3														
4	13														
12	5														
6	11														
10	7														
8	9														
5	12														
11	6														
7	10														
9	8														

KEY TO THE SCHEME OF ALTERNATION WITH QUATERNATE TRANSPOSITION.

A_1	B_{16}	C_1	D_{16}	E_1	F_{16}	G_1	H_{16}	I_1	K_{16}	L_1	M_{16}	N_1	O_{16}	P_1	Q_{16}
Q_{15}	P_2	O_{15}	N_2	M_{15}	L_2	K_{15}	I_2	H_{15}	G_2	F_{15}	E_2	D_{15}	C_2	B_{15}	A_2
A_3	B_{14}	C_3	D_{14}	E_3	F_{14}	G_3	H_{14}	I_3	K_{14}	L_3	M_{14}	N_3	O_{14}	P_3	Q_{14}
Q_{13}	P_4	O_{13}	N_4	M_{13}	L_4	K_{13}	I_4	H_{13}	G_4	F_{13}	E_4	D_{13}	C_4	B_{13}	A_4
A_{16}	B_1	C_{16}	D_1	E_{16}	F_1	G_{16}	H_1	I_{16}	K_1	L_{16}	M_1	N_{16}	O_1	P_{16}	Q_1
Q_2	P_{15}	O_2	N_{15}	M_2	L_{15}	K_2	I_{15}	H_2	G_{15}	F_2	E_{15}	D_2	C_{15}	B_2	A_{15}
A_{14}	B_3	C_{14}	D_3	E_{14}	F_3	G_{14}	H_3	I_{14}	K_3	L_{14}	M_3	N_{14}	O_3	P_{14}	Q_3
Q_4	P_{13}	O_4	N_{13}	M_4	L_{13}	K_4	I_{13}	H_4	G_{13}	F_4	E_{13}	D_4	C_{13}	B_4	A_{13}
A_{12}	B_5	C_{12}	D_5	E_{12}	F_5	G_{12}	H_5	I_{12}	K_5	L_{12}	M_5	N_{12}	O_5	P_{12}	Q_5
Q_6	P_{11}	O_6	N_{11}	M_6	L_{11}	K_6	I_{11}	H_6	G_{11}	F_6	E_{11}	D_6	C_{11}	B_6	A_{11}
A_{10}	B_7	C_{10}	D_7	E_{10}	F_7	G_{10}	H_7	I_{10}	K_7	L_{10}	M_7	N_{10}	O_7	P_{10}	Q_7
Q_8	P_9	O_8	N_9	M_8	L_9	K_8	I_9	H_8	G_9	F_8	E_9	D_8	C_9	B_8	A_9
A_5	B_{12}	C_5	D_{12}	E_5	F_{12}	G_5	H_{12}	I_5	K_{12}	L_5	M_{12}	N_5	O_{12}	P_5	Q_{12}
Q_{11}	P_6	O_{11}	N_6	M_{11}	L_6	K_{11}	I_6	H_{11}	G_6	F_{11}	E_6	D_{11}	C_6	B_{11}	A_6
A_7	B_{10}	C_7	D_{10}	E_7	F_{10}	G_7	H_{10}	I_7	K_{10}	L_7	M_{10}	N_7	O_{10}	P_7	Q_{10}
Q_9	P_8	O_9	N_8	M_9	L_8	K_9	I_8	H_9	G_8	F_9	E_8	D_9	C_8	B_9	A_8

SCHEME OF ALTERNATION WITH QUATERNATE TRANSPOSITION.

1	32	33	64	65	96	97	128	129	160	161	192	193	224	225	256
255	226	223	194	191	162	159	130	127	98	95	66	63	34	31	2
3	30	35	62	67	94	99	126	131	158	163	190	195	222	227	254
253	228	221	196	189	164	157	132	125	100	93	68	61	36	29	4
16	17	48	49	80	81	112	113	144	145	176	177	208	209	240	241
242	239	210	207	178	175	146	143	114	111	82	79	50	47	18	15
14	19	46	51	78	83	110	115	142	147	174	179	206	211	238	243
244	237	212	205	180	173	148	141	116	109	84	77	52	45	20	13
12	21	44	53	76	85	108	117	140	149	172	181	204	213	236	245
246	235	214	203	182	171	150	139	118	107	86	75	54	43	22	11
10	23	42	55	74	87	106	119	138	151	170	183	202	215	234	247
248	233	216	201	184	169	152	137	120	105	88	73	56	41	24	9
5	28	37	60	69	92	101	124	133	156	165	188	197	220	229	252
251	230	219	198	187	166	155	134	123	102	91	70	59	38	27	6
7	26	39	58	71	90	103	122	135	154	167	186	199	218	231	250
249	232	217	200	185	168	153	136	121	104	89	72	57	40	25	8

A SQUARE CONSTRUCTED BY ALTERNATION WITH QUATERNATE TRANSPOSITION.

N_{8}	O_{9}	P_{8}	Q_{9}	A_{8}	B_{9}	C_{8}	D_{9}	E_{8}	F_{9}	G_{8}	H_{9}	I_{8}	K_{9}	L_{8}	M_{9}
D_{10}	C_{7}	B_{10}	A_{7}	Q_{10}	P_{7}	O_{10}	N_{7}	M_{10}	L_{7}	K_{10}	I_{7}	H_{10}	G_{7}	F_{10}	E_{7}
N_{6}	O_{11}	P_{6}	Q_{11}	A_{6}	B_{11}	C_{6}	D_{11}	E_{6}	F_{11}	G_{6}	H_{11}	I_{6}	K_{11}	L_{6}	M_{11}
D_{12}	C_{5}	B_{12}	A_{5}	Q_{12}	P_{5}	O_{12}	N_{5}	M_{12}	L_{5}	K_{12}	I_{5}	H_{12}	G_{5}	F_{12}	E_{5}
N_{9}	O_{8}	P_{9}	Q_{8}	A_{9}	B_{8}	C_{9}	D_{8}	E_{9}	F_{8}	G_{9}	H_{8}	I_{9}	K_{8}	L_{9}	M_{8}
D_{7}	C_{10}	B_{7}	A_{10}	Q_{7}	P_{10}	O_{7}	N_{10}	M_{7}	L_{10}	K_{7}	I_{10}	H_{7}	G_{10}	F_{7}	E_{10}
N_{11}	O_{6}	P_{11}	Q_{6}	A_{11}	B_{6}	C_{11}	D_{6}	E_{11}	F_{6}	G_{11}	H_{6}	I_{11}	K_{6}	L_{11}	M_{6}
D_{5}	C_{12}	B_{5}	A_{12}	Q_{5}	P_{12}	O_{5}	N_{12}	M_{5}	L_{12}	K_{5}	I_{12}	H_{5}	G_{12}	F_{5}	E_{12}
N_{13}	O_{4}	P_{13}	Q_{4}	A_{13}	B_{4}	C_{13}	D_{4}	E_{13}	F_{4}	G_{13}	H_{4}	I_{13}	K_{4}	L_{13}	M_{4}
D_{3}	C_{14}	B_{3}	A_{14}	Q_{3}	P_{14}	O_{3}	N_{14}	M_{3}	L_{14}	K_{3}	I_{14}	H_{3}	G_{14}	F_{3}	E_{14}
N_{15}	O_{2}	P_{15}	Q_{2}	A_{15}	B_{2}	C_{15}	D_{2}	E_{15}	F_{2}	G_{15}	H_{2}	I_{15}	K_{2}	L_{15}	M_{2}
D_{1}	C_{16}	B_{1}	A_{16}	Q_{1}	P_{16}	O_{1}	N_{16}	M_{1}	L_{16}	K_{1}	I_{16}	H_{1}	G_{16}	F_{1}	E_{16}
N_{4}	O_{13}	P_{4}	Q_{13}	A_{4}	B_{13}	C_{4}	D_{13}	E_{4}	F_{13}	G_{4}	H_{13}	I_{4}	K_{13}	L_{4}	M_{13}
D_{14}	C_{3}	B_{14}	A_{3}	Q_{14}	P_{3}	O_{14}	N_{3}	M_{14}	L_{3}	K_{14}	I_{3}	H_{14}	G_{3}	F_{14}	E_{3}
N_{2}	O_{15}	P_{2}	Q_{15}	A_{2}	B_{15}	C_{2}	D_{15}	E_{2}	F_{15}	G_{2}	H_{15}	I_{2}	K_{15}	L_{2}	M_{15}
D_{16}	C_{1}	B_{16}	A_{1}	Q_{16}	P_{1}	O_{16}	N_{1}	M_{16}	L_{1}	K_{16}	I_{1}	H_{16}	G_{1}	F_{16}	E_{1}

SCHEME OF FRANKLIN'S 16×16 SQUARE.

the second quarter to the bottom, we have the key to the entire arrangement of figures; in which the alternation starts as in the scheme for simple alternation but skips the four center rows passing from 2 in the second cell of the second row to 3 in the first cell of the seventh, and from 4 in the second cell of the eighth passing to 5 in the first cell, and thence upwards in similar alternation, again passing over the four central rows to the second and ending with 8 in the second cell of the first row. Then the same alternation is produced in the four center rows. It is obvious that this can not start in the first cell as that would duplicate the first row, so we start with 1 in the second cell passing down uninterruptedly to 4 and ascending as before from 5 to 8.

A closer examination will show that the rows are binate, which means in sets of two. The four inner numbers, 3, 4, 5, 6 and the two outer sets of two numbers each, 1, 2 and 7, 8, are brought together thus imparting to the whole square a binate character.

We are now provided with a key to build up a magic square after the pattern of Franklin. We have simply to complete it in the same way as our last square repeating the letters with their order alternately reversed as before, and repeating the figures in each line.

When we insert their figure values we have a square which is not the same as Franklin's, but possesses in principle the same qualities.

To make our 8×8 square of binate transposition into the Franklin square we must first take its obverse square; that is to say, we preserve exactly the same order but holding the paper with the figures toward the light we read them off from the obverse side, and then take the mirror picture of the result, holding the mirror on either horizontal side. So far we have still our square with the peculiarities of our scheme, but which lacks one of the incidental characteristics of Franklin's square. We must notice that he makes four cells in both horizontal and vertical directions sum up to 130 which property is necessarily limited only to two sets of four cells in each row. If we write down the sum of $1+2+3+4+5+6+7+8=2\times 18$, we will find that the middle set $3+4+5+6$ is equal to the rest consisting of the sum of two extremes, $1+4$, and $7+8$. In this way we cut out in our scheme of symbols, the rows represented by the letters C, D, E, F in either order and accordingly we can shift either of the two first or two last vertical rows to the other side. Franklin did the former, thus beginning

his square with G_4 in the left upper corner. We have indicated this division by heavier lines in both schemes.

The greater square of Franklin, which is 16×16, is made after the same fashion, and the adjoined diagrams will sufficiently explain its construction.

We do not know the method employed by Franklin; we possess only the result, but it is not probable that he derived his square according to the scheme employed here.

Our 16×16 square is not exactly the same as the square of Franklin, but it belongs to the same class. Our method gives the key to the construction, and it is understood that the system here represented will allow us to construct many more squares by simply pushing the square beyond its limits into the opposite row which by this move has to be transferred.

There is the same relation between Franklin's 16×16 square and our square constructed by alternation with quaternate transposition, that exists between the corresponding 8×8 squares.

* * *

Mr. C. A. Browne, Jr. furnishes another interesting square of 27×27 representing in addition to its arithmetical qualities commonly possessed by magic squares some ulterior significance of our calendar system referring to the days of the month as well as the days of the year and cycles of years. It is wonderful, and at first sight mystifying, to observe how the course of nature reflects even to intricate details the intrinsic harmony of mathematical relations; and yet when we consider that nature and pure thought are simply the result of conditions first laid down and then consistently carried out in definite functions of a distinct and stable character, we will no longer be puzzled but understand why science is possible, why man's reason contains the clue to many problems of nature and, generally speaking, why reason with all its wealth of *a priori* thoughts can develop at all in a world that at first sight seems to be a mere chaos of particular facts. The purely formal relations of mathematics, materially considered mere nonentities, constitute the bond of union which encompasses the universe, stars as well as motes, the motions of the Milky Way not less than the minute combinations of chemical atoms, and also the construction of pure thought in man's mind.

Mr. Browne's square is of great interest to Greek scholars because it throws light on an obscure passage in Plato's Republic, re-

ferring to a magic square the center of which is 365, the number of days in a year.

The construction of Mr. Browne's square is based upon the simplest square of odd numbers which is 3×3. But it becomes somewhat complicated by being extended to three in the third power which is 27. Odd magic squares are, as we remember, built up by a progression in staircase fashion, but since those numbers

352	381	326	439	468	413	274	303	248	613	642	587	700	729	674	535	564	509	118	147	92	205	234	179	40	69	14
327	353	379	414	440	466	249	275	301	588	614	640	675	701	727	510	536	562	93	119	145	180	206	232	15	41	67
380	325	354	467	412	441	302	247	276	641	586	615	728	673	702	563	508	537	146	91	120	233	178	207	68	13	42
277	306	251	355	384	329	433	462	407	538	567	512	616	645	590	694	723	668	43	72	17	121	150	95	199	228	173
252	278	304	330	356	382	408	434	460	513	539	565	591	617	643	669	695	721	18	44	70	96	122	148	174	200	226
305	250	279	383	328	357	461	406	435	566	511	540	644	589	618	722	667	696	71	16	45	149	94	123	227	172	201
436	465	410	271	300	245	358	387	332	697	726	671	532	561	506	619	648	593	202	231	176	37	66	11	124	153	98
411	437	463	246	272	298	333	359	385	672	698	724	507	533	559	594	620	646	177	203	229	12	38	64	99	125	151
464	409	438	299	244	273	386	331	360	725	670	699	560	505	534	647	592	621	230	175	204	65	10	39	152	97	126
127	156	101	214	243	188	49	78	23	361	390	335	448	477	422	283	312	257	595	624	569	682	711	656	517	546	491
102	128	154	189	215	241	24	50	76	336	362	388	423	449	475	258	284	310	570	596	622	657	683	709	492	518	544
155	100	129	242	187	216	77	22	51	389	334	363	476	421	450	311	256	285	623	568	597	710	655	684	545	490	519
52	81	26	130	159	104	208	237	182	286	315	260	364	393	338	442	471	416	520	549	494	598	627	572	676	705	650
27	53	79	105	131	157	183	209	235	261	287	313	339	365	391	417	443	469	495	521	547	573	599	625	651	677	703
80	25	54	158	103	132	236	181	210	314	259	288	392	337	366	470	415	444	548	493	522	626	571	600	704	649	678
211	240	185	46	75	20	133	162	107	445	474	419	280	309	254	367	396	341	679	708	653	514	543	488	601	630	575
186	212	238	21	47	73	108	134	160	420	446	472	255	281	307	342	368	394	654	680	706	489	515	541	576	602	628
239	184	213	74	19	48	161	106	135	473	418	447	308	253	282	395	340	369	707	652	681	542	487	516	629	574	603
604	633	578	691	720	665	526	555	500	109	138	83	196	225	170	31	60	5	370	399	344	457	486	431	292	321	266
579	605	631	666	692	718	501	527	553	84	110	136	171	197	223	6	32	58	345	371	397	432	458	484	267	293	319
632	577	606	719	664	693	554	499	528	137	82	111	224	169	198	59	4	33	398	343	372	485	430	459	320	265	294
529	558	503	607	636	581	685	714	659	34	63	8	112	141	86	190	219	164	295	324	269	373	402	347	451	480	425
504	530	556	582	608	634	660	686	712	9	35	61	87	113	139	165	191	217	270	296	322	348	374	400	426	452	478
557	502	531	635	580	609	713	658	687	62	7	36	140	85	114	218	163	192	323	268	297	401	346	375	479	424	453
688	717	662	523	552	497	610	639	584	193	222	167	28	57	2	115	144	89	454	483	428	289	318	263	376	405	350
663	689	715	498	524	550	585	611	637	168	194	220	3	29	55	90	116	142	429	455	481	264	290	316	351	377	403
716	661	690	551	496	525	638	583	612	221	166	195	56	1	30	143	88	117	482	427	456	317	262	291	404	349	378

THE MAGIC SQUARE OF MR. C. A. BROWNE.

that fall outside the square have to be transferred to their corresponding places inside, the first and last staircases are changed into the knight's move of the chessboard, and only the middle one retains its original staircase form. We must construct the square so that the central figure, which in a 3×3 square is 5, must always fall in the central cell. Accordingly, we must start the square

beginning with figure 1 outside of the square in any middle cell immediately bordering upon it, which gives four starting-points from which we may either proceed from the right or the left, either upwards or downwards which yields eight possibilities of the 3×3 square. For the construction of his 27×27 square, Mr. Browne might have taken any of these eight possibilities as his pattern. He selected the one starting on the top of the square and moving toward the right, and thus he always follows the peculiar arrangement of this particular square. It is the fourth in our adjoined diagrams of the 3×3 squares. Any one who will take the trouble to trace the regular succession of Mr. Browne's square will find that it is a constant repetition of the knight's move, the staircase move

		3		
	2	7	6	
1	9	5	1	9
	4	3	8	
		7		

		7		
	4	3	8	
1	9	5	1	9
	2	7	6	
		3		

		1		
	2	9	4	
3	7	5	3	7
	6	1	8	
		9		

		1		
	4	9	2	
7	3	5	7	3
	8	1	6	
		9		

		7		
	8	3	4	
9	1	5	9	1
	6	7	2	
		3		

		3		
	6	7	2	
9	1	5	9	1
	8	3	4	
		7		

		9		
	6	1	8	
3	7	5	3	7
	2	9	4	
		1		

		9		
	8	1	6	
7	3	5	7	3
	4	9	2	
		1		

THE EIGHT POSSIBLE ARRANGEMENTS OF THE 3×3 MAGIC SQUARE.

and again a knight's move on a small scale of 3×3 which is repeated on a larger scale 9×9, thus leading to the wonderful regularity which, according to Mr. Browne's interpretation of Plato, astonished the sages of ancient Greece.

Any one who discovers at random some magic square with its immanent harmony of numbers, is naturally impressed by its apparent occult power, and so it happens that they were deemed supernatural and have been called "magic." They seem to be the product of some secret intelligence and to contain a message of ulterior meaning. But if we have the key to their regularity we know that the harmony that pervades them is necessary and intrinsic.

Nor is the regularity limited to magic squares. There are other number combinations which exhibit surprising qualities, and I will here select a few striking cases.

If we write down all the nine figures in ascending and descending order we have a number which is equal to the square of a number consisting of the figure 9 repeated 9 times, divided by the sum of an ascending and descending series of all the figures thus:

$$12345678987654321 = \frac{999999999 \times 999999999}{1+2+3+4+5+6+7+8+9+8+7+6+5+4+3+2+1}.$$

The secret of this mysterious coincidence is that $11 \times 11 = 121$; $111 \times 111 = 12321$; $1111 \times 1111 = 1234321$, etc., and a sum of an ascending or a descending series which starts with 1 is always equal to the square of its highest number. $1+2+1 = 2 \times 2$; $1+2+3+4+3+2+1 = 4 \times 4$, etc., which we will illustrate by one more instance of the same kind, as follows:

$$1234567654321 = \frac{7777777 \times 7777777}{1+2+3+4+5+6+7+6+5+4+3+2+1}.$$

There are more instances of numerical regularities.

All numbers consisting of six equal figures are divisible by 7, and also, as a matter of course, by 3 and 11, as indicated in the following list:

$$\begin{aligned}
111111 : 7 &= 15873\\
222222 : 7 &= 31746\\
333333 : 7 &= 47619\\
444444 : 7 &= 63492\\
555555 : 7 &= 79365\\
666666 : 7 &= 95238\\
777777 : 7 &= 111111\\
888888 : 7 &= 126984\\
999999 : 7 &= 142857
\end{aligned}$$

Finally we will offer two more strange coincidences of a *lusus numerorum*.

$$\begin{aligned}
0 \times 9 + 1 &= 1\\
1 \times 9 + 2 &= 11\\
12 \times 9 + 3 &= 111\\
123 \times 9 + 4 &= 1111\\
1234 \times 9 + 5 &= 11111\\
12345 \times 9 + 6 &= 111111\\
123456 \times 9 + 7 &= 1111111\\
1234567 \times 9 + 8 &= 11111111\\
12345678 \times 9 + 9 &= 111111111\\
123456789 \times 9 + 10 &= 1111111111.
\end{aligned}$$

$$1\times8+1=9$$
$$12\times8+2=98$$
$$123\times8+3=987$$
$$1234\times8+4=9876$$
$$12345\times8+5=98765$$
$$123456\times8+6=987654$$
$$1234567\times8+7=9876543$$
$$12345678\times8+8=98765432$$
$$123456789\times8+9=987654321.$$

No wonder that such strange regularities impress the human mind. A man who knows only the externality of these results will naturally be inclined toward occultism. The world of numbers as much as the actual universe is full of regularities which can be reduced to definite rules and laws giving us a key that will unlock their mysteries and enable us to predict certain results under definite conditions. Here is the key to the significance of the *a priori.*

Mathematics is a purely mental construction, but its composition is not arbitrary. On the contrary it is tracing the results of our own doings and taking the consequences of the conditions we have created. Though the scope of our imagination with all its possibilities be infinite, the results of our construction are definitely determined as soon as we have laid their foundation, and the actual world is simply one realization of the infinite potentialities of being. Its regularities can be unraveled as surely as the harmonic relations of a magic square.

Facts are just as much determined as our thoughts, and if we can but gain a clue to their formation we can solve the problem of their nature, and are enabled to predict their occurrence and sometimes even to adapt them to our own needs and purposes.

A study of magic squares may have no practical application, but an acquaintance with them will certainly prove useful, if it were merely to gain an insight into the fabric of regularities of any kind.

EDITOR.